

A Coassociative C^* -Quantum Group with Non-Integral Dimensions

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Abstract

By weakening the counit and antipode axioms of a C^* -Hopf algebra and allowing for the coassociative coproduct to be non-unital we obtain a quantum group, that we call a *weak C^* -Hopf algebra*, which is sufficiently general to describe the symmetries of essentially arbitrary fusion rules. This amounts to generalizing the Baaj-Skandalis multiplicative unitaries to multiplicative partial isometries. Every weak C^* -Hopf algebra has a dual which is again a weak C^* -Hopf algebra. An explicit example is presented with Lee-Yang fusion rules. We shortly discuss applications to amalgamated crossed products, doubles, and quantum chains.

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1 Introduction

Conformal field theories provide examples of quantum field theory models with finitely many superselection sectors p, q, \dots such that their intrinsic dimensions d_p can take non-integer values. Here d_p denotes a component of the Perron–Frobenius eigenvector of the fusion matrices, $\sum_r N_{pq}^r d_r = d_p d_q$, but also coincides with the statistical dimension in the sense of Algebraic QFT [H]. The symmetry of such a QFT cannot be described by a finite dimensional C^* -Hopf algebra H . As a matter of fact if H is isomorphic to $\oplus_r \text{Mat}(n_r, \mathbb{C})$ then $d_p = n_p$ solves the above equation as a consequence of the unitalness of the coproduct: $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$. In order to fit to the new situation, the concept of a Hopf algebra has been replaced by more and more general structures: quasi-Hopf algebras [Dr], truncated (or weak) quasi-Hopf algebras [MS2], and rational Hopf algebras [V, FGV1].

Although the rational Hopf algebra approach successfully reproduces all data of the quantum field theory encoded in the representation category $\mathbf{Rep} \mathcal{A}$ of its observable algebra \mathcal{A} , it has a serious flaw: Its dual is not an associative algebra, therefore "crossed product" of \mathcal{A} with a coaction of the symmetry, which is usually the algebra \mathcal{F} of charge carrying fields, is non-associative. The same non-associativity forbids also to define an action of the symmetry on the fields without introducing at the same time operators implementing the whole symmetry algebra [MS2]. In the theory of quantum chains one faces the analogue problem if one would like to construct a quantum chain with non-integer statistical dimensions. The strategy of the construction [NSz1, NSz2] requires the observable algebra to be an iterated crossed product $\dots \rtimes A \rtimes \hat{A} \rtimes A \rtimes \dots$ of a Hopf-like algebra A and its dual \hat{A} . This would turn the observable algebra to be non-associative as well as its "symmetry", the "double" of A .

These problems naturally raise the question: Was it necessary to give up coassociativity in order to cover all interesting cases of non-integral fusion rules? Our proposal of a *weak C^* -Hopf algebra* shows that coassociativity can be maintained even in the most general case. Its axioms will be given in Sect. 2. It is a C^* -algebra A and in applications to rational quantum field theory models it is finite dimensional. In the interesting cases it does not have, however, any 1-dimensional representation. Therefore the counit $\varepsilon: A \rightarrow \mathbb{C}$ can only be a coalgebraic counit and not an algebra map any more. The coproduct $A \ni x \mapsto \Delta(x) \equiv x_{(1)} \otimes x_{(2)}$ is coassociative but not unit preserving in general. The antipode axiom is weakened as well so that the maps $A \ni x \mapsto S(x_{(1)})x_{(2)}$ and $A \ni x \mapsto x_{(1)}S(x_{(2)})$ become projections Π^R and Π^L , respectively, to certain non-trivial C^* -subalgebras of A that play important role in crossed products: These subalgebras will give amalgamations (see Sect. 3) between A and its dual \hat{A} in the Weyl algebras $A \rtimes \hat{A}$, $\hat{A} \rtimes A$, and in the double $\mathcal{D} = A \bowtie \hat{A}$.

It is important to emphasize that our axioms define a selfdual algebraic structure, just like the axioms for a Hopf algebra. That is the dual \hat{A} can be given structural maps $\hat{\Delta}, \hat{\varepsilon}, \hat{S}$ satisfying the same axioms as Δ, ε , and S do for A .

The simplest example for such a weak Hopf algebra can be found [see Appendix] in studying the quasi-double $\mathcal{D}^\omega(G)$ of a finite group G [DPR]. $\mathcal{D}^\omega(G)$ has a 1-dimensional block corresponding to the trivial representation but its coproduct is quasi coassociative. "Blowing up" the double by a full matrix algebra M_n , that is introducing the algebra $\mathcal{M}^\omega(G) := \mathcal{D}^\omega(G) \otimes M_n$, where n can be taken to be the order of G , there is a coassociative coproduct on $\mathcal{M}^\omega(G)$ producing precisely the fusion rules N_{pq}^r of $\mathcal{D}^\omega(G)$. This coassociative coproduct is related to the quasi-coassociative coproduct of $\mathcal{D}^\omega(G)$ by a "skrooching" (a name proposed by J. Stasheff [St]), i.e. by conjugation with a partial isometry $U \in \mathcal{M}^\omega(G) \otimes \mathcal{M}^\omega(G)$. Transforming out a non-trivial cocycle ω by skrooching was not possible in $\mathcal{D}^\omega(G)$ but there is more flexibility in the blown up version $\mathcal{M}^\omega(G)$. Not only their fusion rules coincide but also the representation

category of $\mathcal{M}^\omega(G)$ is equivalent to that of the quasi double. It turns out that $\mathcal{M}^\omega(G)$ itself is a kind of double. It is the double of a blown up version of the Hopf algebra of complex functions on G .

The dimensions of $\mathcal{M}^\omega(G)$ are of course integers. In order to find examples with non-integer dimensions one needs to construct weak C^* -Hopf algebras A with $S^2 \neq \text{id}$. Namely, one can prove that the dimensions d_p of a weak C^* -Hopf algebra are all integers if the antipode is involutive. (Recall that in case of compact matrix pseudogroups [W] $S^2 = \text{id}$ follows if A is finite dimensional. Therefore non-integer dimensions can be expected only for infinite dimensional compact matrix pseudogroups.) There is a general construction of finite dimensional weak C^* -Hopf algebras from a solution F of the pentagon equation, typically having non-integral d_p -s and therefore $S^2 \neq \text{id}$. One starts from a given set of fusion matrices N_{pq}^r ³ and a corresponding solution $F^{(pqr)}_{stu}$ of the pentagon equations

$$\sum_e F^{(pqr)}_{se} F^{(aer)}_{ds} F^{(apq)}_{ce} = F^{(apf)}_{ds} F^{(bqr)}_{cf} \quad (1.1)$$

and the unitarity conditions

$$\sum_e \overline{F^{(pqr)}_{se}} F^{(pqr)}_{eg} = \delta_{fg} N_{qr}^g N_{pg}^s. \quad (1.2)$$

Such a solution is provided in any quantum field theory with finitely many superselection sectors as it was emphasized in [MS1, S]. In spite of its tempting similarity to the pentagon equation for the reassociator φ of a quasi-Hopf algebra [Dr] triviality of φ is not the same as triviality of F . Hence a coassociative quantum symmetry (trivial φ) does not imply any kind of simplification on the solution F . Quite on the contrary, we will see in Sect.4 that eqn. (1.1) can always be read as the coassociativity condition of an appropriate coproduct.

The weak C^* -Hopf algebra A we construct from (1.1–2) has fusion rules precisely given by N_{pq}^r and its $6j$ -symbols by $F^{(pqr)}_{stu}$. As a C^* -algebra it is isomorphic to $\oplus_p M_{n_p}$ where $n_p = \sum_{ab} N_{ap}^b$. For example in case of the Lee-Yang fusion rules one obtains (see Sect.5) the algebra $A = M_2 \oplus M_3$ which can be thought of an inhomogeneous "blowing up" of the quasi-coassociative rational Hopf algebra $M_1 \oplus M_2$ of [FGV1]. For the Ising fusion rules our construction yields $A = M_3 \oplus M_3 \oplus M_4$ instead of the quasi-coassociative $M_1 \oplus M_1 \oplus M_2$.

We briefly mention that weak C^* -Hopf algebras can also be described by the methods of Baaj and Skandalis [BS]. Let \mathcal{H} be a Hilbert space and V a partial isometry acting on $\mathcal{H} \otimes \mathcal{H}$. We call V a *multiplicative partial isometry* if the following two equations hold true on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$.

$$\begin{aligned} V_{12} V_{13} V_{23} &= V_{23} V_{12} \\ V_{13} V_{23} V_{23}^* &= V_{12}^* V_{12} V_{13} \end{aligned}$$

To this data there is an associated pair (A, \hat{A}) of weak C^* -Hopf algebras in duality. The reverse statement can be verified easily. Namely, if A is a weak C^* -Hopf algebra then \hat{A} is dense in the Hilbert space \mathcal{H} carrying the left regular representation of A and

$$V(x \otimes y) := x_{(1)} \otimes x_{(2)} y \quad x, y \in A \subset \mathcal{H}$$

defines a multiplicative partial isometry. Its initial and final projections are $V^*V = \Delta(\mathbb{1})$ and $VV^* = \hat{\Delta}(\hat{\mathbb{1}})$, respectively.

Most of our results refer, at present, to finite dimensional weak C^* -Hopf algebras although there seems to be no obstruction towards a generalization to the infinite dimensional case.

Ideas about a coassociative C^* -quantum group very similar to ours have been proposed earlier by F. Nill (unpublished).

³Here we restrict ourselves to multiplicity free fusion rules: $N_{pq}^r \leq 1$.

2 Axioms of Weak C^* -Hopf Algebras

Definition 2.1 A weak $*$ -Hopf algebra is a $*$ -algebra A with unit $\mathbb{1}$ together with linear maps $\Delta: A \rightarrow A \otimes A$, $\varepsilon: A \rightarrow \mathbb{C}$, and $S: A \rightarrow A$ called the coproduct, the counit, and the antipode respectively, if the following axioms hold:

$$\Delta(xy) = \Delta(x)\Delta(y) \quad (\text{A.1a})$$

$$\Delta(x^*) = \Delta(x)^* \quad (\text{A.1b})$$

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \quad (\text{A.1c})$$

$$\varepsilon(xy) = \varepsilon(x\mathbb{1}_{(1)})\varepsilon(\mathbb{1}_{(2)}y) \quad (\text{A.2a})$$

$$\varepsilon(x^*x) \geq 0 \quad (\text{A.2b})$$

$$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta \quad (\text{A.2c})$$

$$S(xy) = S(y)S(x) \quad (\text{A.3a})$$

$$S \circ^* \circ S \circ^* = id \quad (\text{A.3b})$$

$$\Delta \circ S = (S \otimes S) \circ \Delta^{op} \quad (\text{A.3c})$$

$$S(x_{(1)}x_{(2)}) \otimes x_{(3)} = \mathbb{1}_{(1)} \otimes x\mathbb{1}_{(2)} \quad (\text{A.4})$$

for all $x, y \in A$. If furthermore A possesses a faithful $*$ -representation on a Hilbert space it is called a weak C^* -Hopf algebra.

If we add to these axioms either the condition $\Delta(\mathbb{1}) = \mathbb{1} \otimes \mathbb{1}$, or the condition $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$, or the condition $S(x_{(1)})x_{(2)} = \mathbb{1}\varepsilon(x)$ then A becomes a usual $(C)^*$ -Hopf algebra. We do not know whether a simpler set of axioms exists that are equivalent to (A.1–4). But our axioms go definitely beyond Hopf algebras by the vast class of examples with non-integer dimensions constructed in Sect. 4.

Now we give, without proofs, some of the consequences of the above axioms. Details will be published elsewhere [BNSz]. The maps $\Pi^{L/R}$ defined by

$$\Pi^L(x) := x_{(1)}S(x_{(2)}), \quad \Pi^R(x) := S(x_{(1)})x_{(2)} \quad (2.1)$$

are not conditional expectations, are not even $*$ -preserving, but are linear projections that project onto C^* -subalgebras A^L , resp. A^R of A . They are isomorphic as C^* -algebras, the antipode maps A^L onto A^R , A^L commutes with A^R , and the restriction of S^2 onto $A^{L/R}$ is the identity. The restriction of the counit onto A^L (or A^R) is a faithful trace. We may then define an orthonormal basis $\{e^i\}$ of A^L , i.e. $\varepsilon(e^{i*}e^j) = \delta^{i,j}$. It follows that $e_i := S(e^{i*})$ defines an orthonormal basis for A^R . The coproduct of the unit can be expressed as $\Delta(\mathbb{1}) = e_i \otimes e^i$ (summation over i is understood). It follows that $N := \varepsilon(\mathbb{1})$ is a positive integer equal to the dimension of $A^{L/R}$.

The dual \hat{A} of A is defined to be the space of linear functionals φ on A and is equipped with a multiplication and a comultiplication obtained by transposing the comultiplication and

multiplication of A w.r.t the canonical pairing $\langle \cdot, \cdot \rangle: \hat{A} \times A \rightarrow \mathbb{C}$. The unit element of \hat{A} is $\hat{\mathbb{1}} := \varepsilon$. The antipode \hat{S} and the star operation of \hat{A} are defined respectively by

$$\langle \hat{S}(\varphi), x \rangle = \langle \varphi, S(x) \rangle \quad (2.2)$$

$$\langle \varphi^*, x \rangle = \overline{\langle \varphi, S(x)^* \rangle} \quad (2.3)$$

Theorem 2.2 *If $(A, \mathbb{1}, \Delta, \varepsilon, S)$ satisfies Axioms (A1–4) then $(\hat{A}, \hat{\mathbb{1}}, \hat{\Delta}, \hat{\varepsilon}, \hat{S})$ satisfies Axioms (A1–4), too. That is the notion of a weak * -Hopf algebra is selfdual.*

The proof requires a little work because our axioms are not so transparently selfdual as those of a Hopf algebra. For example in order to verify the crucial axioms (A.2a) and (A.4) in the dual one proves at first that the relations

$$\begin{aligned} \mathbb{1}_{(1)} \otimes \varepsilon(\mathbb{1}_{(2)} \mathbb{1}_{(1)'}) \mathbb{1}_{(2)'} &= \mathbb{1}_{(1)} \otimes \mathbb{1}_{(2)} \\ S(x_{(1)}) x_{(2)} y &= y_{(1)} \varepsilon(x y_{(2)}) \quad x, y \in A \end{aligned}$$

are consequences of (A1–4). Then pairing the first with $\varphi \otimes \psi \in \hat{A} \otimes \hat{A}$ and the second with $\varphi \in \hat{A}$ one obtains the dual counterparts of (A.2a) and (A.4), respectively.

Just like in case of a pair of dual Hopf algebras there are natural left and right actions of A on \hat{A} given by the Sweedler's arrows

$$x \rightarrow \varphi := \varphi_{(1)} \langle \varphi_{(2)}, x \rangle, \quad \varphi \leftarrow x := \langle \varphi_{(1)}, x \rangle \varphi_{(2)} \quad (2.4)$$

for all $\varphi \in \hat{A}$ and $x \in A$. Similarly one defines the left and right actions of \hat{A} on A .

The canonical pairing $\langle \cdot, \cdot \rangle$ restricted to $\hat{A}^L \times A^R$ is non-degenerate and so is its restriction to $\hat{A}^R \times A^L$. The bases dual to $\{e_i\}$ and $\{e^i\}$ are found to be $\hat{e}^i = \hat{\mathbb{1}} \leftarrow e^i \in \hat{A}^L$ and $\hat{e}_i = e_i \rightarrow \hat{\mathbb{1}} \in \hat{A}^R$, respectively.

From now on we assume that A satisfies (A.1–4) and is a finite dimensional C^* -algebra ("finite quantum group"). Let $\mathbf{Rep} A$ denote the category of finite dimensional (not necessarily non-degenerate) * -representations of A . Its arrows are the intertwiners $T: D_1 \rightarrow D_2$ with the unit arrow at the object D being $\mathbf{1}_D = D(\mathbb{1})$. $\mathbf{Rep} A$ becomes a monoidal category if we define for objects D_1, D_2 the product representation by $D_1 \times D_2 := (D_1 \otimes D_2) \circ \Delta$ and for arrows $T: D_1 \rightarrow D_2, T': D'_1 \rightarrow D'_2$ the product arrow by $T \times T' := (T \otimes T') \cdot (D_1 \times D'_1)(\mathbb{1})$. The unit object (i.e. the trivial representation of A) is defined to be the GNS representation D_ε associated to the state $\frac{1}{N} \cdot \varepsilon$. Using the identity

$$\varepsilon(x^* y) = \varepsilon(\Pi^L(x)^* \Pi^L(y)) \quad (2.5)$$

one recognizes that the representation space of D_ε is A^L equipped with the Hilbert-Schmidt scalar product $(x^L, y^L) := \varepsilon(x^L {}^* y^L)$, $x^L, y^L \in A^L$. The action is given by $D_\varepsilon(x) \Pi^L(y) = \Pi^L(xy)$, $x, y \in A$. Sometimes it is convenient to use matrix elements of D_ε in the orthonormal basis $\{e^{i*}\}$ of A^L : $D_\varepsilon^{ij}(x) = \varepsilon(e^i x e_j)$. The name trivial representation for D_ε is justified by the existence of isometric intertwiners $u_D^L: D \rightarrow D_\varepsilon \times D$ and $u_D^R: D \rightarrow D \times D_\varepsilon$ that satisfy the triangle identities

$$u_{D_1 \times D_2}^L = u_{D_1}^L \times \mathbf{1}_{D_2} \quad (2.6a)$$

$$\mathbf{1}_{D_1} \times u_{D_2}^L = u_{D_1}^R \times \mathbf{1}_{D_2} \quad (2.6b)$$

$$u_{D_1 \times D_2}^R = \mathbf{1}_{D_1} \times u_{D_2}^R \quad (2.6c)$$

and are natural in D . They can be defined by the matrix elements

$$(u_D^L)^{i\alpha,\beta} := D^{\alpha\beta}(e^i) \quad (2.7)$$

$$(u_D^R)^{\alpha i,\beta} := D^{\alpha\beta}(e_i^*) \quad (2.8)$$

The usual formula $\bar{D} = D^T \circ S$ for the conjugate representation does not work unless $S^2 = \text{id}$, since \bar{D} is not unitary in general. However, there exists an $S_0 \in \text{End } A$ and an invertible $C \in A$ such that $S_0^2 = \text{id}$, $S_0 \circ * = * \circ S_0$, and $S = \text{Ad}_C \circ S_0$. Then we may define the conjugate of the representation D by

$$\bar{D}^{\alpha\beta}(x) := D^{\beta\alpha}(S_0(x)) \quad x \in A. \quad (2.9)$$

Rigidity intertwiners can be obtained as follows. Let

$$(R_D)^{\bar{\alpha}\alpha,i} := D^{\alpha\bar{\alpha}}(e_i S_0(C)) \quad (2.10)$$

then R_D is an injective intertwiner from D_ε to $\bar{D} \times D$. Furthermore let $\bar{R}_D := R_{\bar{D}}$. Then the rigidity relations

$$u_D^{L*}(\bar{R}_D^* \times \mathbf{1}_D)(\mathbf{1}_D \times R_D)u_D^R = \mathbf{1}_D \quad (2.11)$$

$$u_D^{R*}(\mathbf{1}_{\bar{D}} \times \bar{R}_D^*)(R_D \times \mathbf{1}_{\bar{D}})u_D^L = \mathbf{1}_{\bar{D}} \quad (2.12)$$

hold provided we have adjusted C to satisfy $S(C^*)C = \mathbb{1}$, which is always possible in view of the freedom in multiplying C by a central invertible element. We have therefore

Theorem 2.3 *If A is a finite dimensional weak C^* -Hopf algebra then the category $\mathbf{Rep } A$ of its finite dimensional $*$ -representations is a rigid monoidal C^* -category.*

Coassociativity of the coproduct leads to "almost" strict monoidality of $\mathbf{Rep } A$. Although the reassociator maps $\varphi_{D_1,D_2,D_3}: (D_1 \times D_2) \times D_3 \rightarrow D_1 \times (D_2 \times D_3)$ are the trivial identity arrows one needs the non-trivial isometric arrows u_D^L and u_D^R to compare D with $D_\varepsilon \times D$ or $D \times D_\varepsilon$.

Somewhat surprisingly the trivial representation need not be irreducible. Consider the example where A is a finite dimensional Abelian C^* -algebra with 1-dimensional irreducible representations D_p obeying the fusion rules $D_p \times D_q = \delta_{p,q} D_q$. Now it is clear that the "trivial" representation is not only reducible but even faithful: $D_\varepsilon = \sum_p D_p$. However, this quantum group is a sum of (1-dimensional) quantum groups each of them having an irreducible trivial representation. This is a general phenomenon as the next Proposition shows.

Proposition 2.4 *For A a finite dimensional weak C^* -Hopf algebra let \mathcal{S} denote the set of its irreducible sectors and let $D_\varepsilon = \bigoplus_{p \in \mathcal{S}} \bigoplus_{\mu=1}^{\nu_p} D_p$ be the decomposition of the trivial representation into irreducibles.*

Then $\nu_p \leq 1$, i.e. D_ε is multiplicity free, $\bar{p} = p$ for all $p \in \mathcal{S}$ such that $\nu_p = 1$, i.e. the trivial representation contains only selfconjugate irreducibles. Let $\mathcal{Z} := \{p \in \mathcal{S} | \nu_p = 1\}$ and for $p \in \mathcal{Z}$ define the subsets of sectors

$$\mathcal{S}_p := \{q \in \mathcal{S} | N_{pq}^q = 1\}.$$

Then $\{\mathcal{S}_p | p \in \mathcal{Z}\}$ is a partition of \mathcal{S} each member of which is closed under the monoidal product (i.e. $p \in \mathcal{Z}$, $q_1, q_2 \in \mathcal{S}_p$, and $N_{q_1 q_2}^r > 0$ imply that $r \in \mathcal{S}_p$) and under conjugation ($p \in \mathcal{Z}$, $q \in \mathcal{S}_p$ imply $\bar{q} \in \mathcal{S}_p$). The irreducible D_p serves as a monoidal unit for the full subcategory $\mathbf{Rep}_p A$ of $\mathbf{Rep } A$ generated by the irreducibles $q \in \mathcal{S}_p$. If $p \neq p'$, the monoidal product of any $D \in \mathbf{Rep}_p A$ and $D' \in \mathbf{Rep}_{p'} A$ is the zero object, $D \times D' = 0$, and the intertwiner space $(D|D')$ is zero dimensional.

Corollary 2.5 *Any finite dimensional weak C^* -Hopf algebra A can be decomposed uniquely into a sum $A = \oplus_p A_p$ of weak C^* -Hopf subalgebras such that the counits $\varepsilon_p = \varepsilon|_{A_p}$ of A_p are pure (unnormalized) states.*

If A has a counit that is pure A will be called *pure*. For pure A the rigidity intertwiners can be normalized to obey

$$R_D^* R_D = d_D \mathbf{1}_{D_\varepsilon}, \quad \bar{R}_D^* \bar{R}_D = d_D \mathbf{1}_{D_\varepsilon} \quad (2.13)$$

with a uniquely determined positive number d_D , called the dimension of D . The function $D \mapsto d_D$ is additive and multiplicative for direct sums and for the monoidal product of representations.

Like compact groups finite dimensional weak C^* -Hopf algebras possess unique Haar measures in the following sense. There exists a unique $h \in A$ characterized by the property

$$xh = \Pi^L(x)h, \quad hx = h\Pi^R(x), \quad \forall x \in A \quad (2.14)$$

and by the normalization conditions $\Pi^L(h) = \mathbf{1} = \Pi^R(h)$. For Hopf algebras $\Pi^{L/R} = \varepsilon$, therefore this definition coincides with the usual one [Sw]. The Haar measure h satisfies the following important properties:

$$\langle \varphi^* \varphi, h \rangle \geq 0 \quad \varphi \in \hat{A} \quad (2.15a)$$

$$\varphi \geq 0, \langle \varphi, h \rangle = 0 \Rightarrow \varphi = 0 \quad (2.15b)$$

$$\langle \hat{\mathbf{1}}, h \rangle = N \quad (2.15c)$$

$$h^2 = h = h^* = S(h) \quad (2.15d)$$

$$h_{(1)}x \otimes h_{(2)} = h_{(1)} \otimes h_{(2)}S(x) \quad x \in A \quad (2.15e)$$

It follows that the Haar integral $\int \varphi dh \equiv \langle \varphi, h \rangle$ is left and right invariant:

$$\langle (x \rightarrow \varphi)\psi, h \rangle = \langle \varphi(S(x) \rightarrow \psi), h \rangle \quad (2.16)$$

$$\langle (\varphi \leftarrow x)\psi, h \rangle = \langle \varphi(\psi \leftarrow S^{-1}(x)), h \rangle \quad (2.17)$$

Furthermore the maps

$$E^L(\varphi) := h \rightarrow \varphi, \quad E^R(\varphi) := \varphi \leftarrow h \quad (2.18)$$

define conditional expectations from \hat{A} onto \hat{A}^L and \hat{A}^R , respectively. In order to show its existence we give an explicit expression for h . Choose the orthonormal basis $\{e^i\}$ to be diagonal with respect to the decomposition $A = \oplus_p A_p$ of Corollary 2.5. Then $D_\varepsilon^{ij} \equiv \hat{e}^i \hat{e}_j$ is zero unless i and j belong to the same block $p \in \mathcal{Z}$. Choose matrix units $\{e_p^{ij}\}$ in the trivial block of A_p that are dual to D_ε^{ij} , i.e. $\langle D_\varepsilon^{ij}, e_p^{kl} \rangle = \delta^{ik} \delta^{jl} \delta_{k \in p} \delta_{l \in p}$. Then

$$h = \sum_{p \in \mathcal{Z}} \frac{1}{N_p} \sum_{i,j \in p} \varepsilon(e^i) e_p^{ij} \varepsilon(e_j) \quad (2.19)$$

and its coproduct takes the following form using matrix units $\{e_r^{\alpha\beta}\}$, $\alpha, \beta = 1, \dots, n_r$ of A :

$$\begin{aligned} \Delta(h) &= (S \otimes \text{id})(X) \cdot (C \otimes C^{-1}) \\ \text{where } X &= \sum_{p \in \mathcal{Z}} \frac{1}{N_p} \sum_{r \in \mathcal{S}_p} \frac{1}{d_r} \sum_{\alpha, \beta=1}^{n_r} e_r^{\alpha\beta} \otimes e_r^{\beta\alpha} = X^* = X^{op}. \end{aligned}$$

Since the existence of the Haar measure in A implies that $\hat{A} \ni \varphi \mapsto \langle \varphi, h \rangle$ is a faithful positive linear functional, \hat{A} is also a C^* -algebra. This shows that selfduality in the sense of Theorem 2.2 is also true for finite dimensional C^* -weak Hopf algebras.

Like in [W] one can prove that the Hermitean invertible element $s := CC^*$ implementing the square of the antipode, $sxs^{-1} = S^2(x)$, $x \in A$, is grouplike, that is $\Delta(s) = (s \otimes s)\Delta(\mathbb{1}) = \Delta(\mathbb{1})(s \otimes s)$. Although our finite quantum group is always unimodular in the sense that it has a 2-sided Haar measure, the modular operator in the left regular representation is non-trivial because the Haar state is not a trace:

$$\langle \varphi\psi, h \rangle = \langle \psi(s \rightarrow \varphi \leftarrow s), h \rangle \quad (2.20)$$

If A is pure the dimension function can be expressed as $d_D = \frac{1}{N} \chi_D(s^{-1})$, where χ_D denotes ordinary (unnormalized) trace in the representation D .

3 The Weyl algebras and the Double

The Weyl algebra $A \bowtie \hat{A}$ of a weak $*$ -Hopf algebra A is defined to be the unital $*$ -algebra generated by A and \hat{A} , as unital $*$ -subalgebras, subjected to the commutation relation

$$\varphi x = x_{(1)} \langle x_{(2)}, \varphi_{(1)} \rangle \varphi_{(2)} \quad (3.1)$$

It follows that the Weyl algebra is a $*$ -algebra in which the Sweedler arrows $\hat{A} \rightarrow A$ and $\hat{A} \leftarrow A$ become inner:

$$\varphi \rightarrow x = \varphi_{(1)} x \hat{S}(\varphi_{(2)}) \quad (3.2)$$

$$\varphi \leftarrow x = S(x_{(1)}) \varphi x_{(2)} \quad (3.3)$$

If A is a Hopf $*$ -algebra then its Weyl algebra is isomorphic to $A \otimes \hat{A}$ as a linear space and is simple as an algebra [N]. In our more general setting $A \bowtie \hat{A}$ is an amalgamated tensor product $A \otimes_{A^R \cong \hat{A}^L} \hat{A}$ over the common $*$ -subalgebra $A^R \cong \hat{A}^L$ identified through the isomorphism $A^R \ni x^R \mapsto (\hat{\mathbb{1}} \leftarrow x^R) \in \hat{A}^L$. In other words $\hat{e}^i = e_i^*$ is an identity in $A \bowtie \hat{A}$ for all $i = 1, \dots, N$. The other two "edge" subalgebras turn out to be the relative commutants of A and \hat{A} within the Weyl algebra, thus we have

$$\begin{aligned} A \cap \hat{A} &= A^R \equiv \hat{A}^L \\ A' \cap (A \bowtie \hat{A}) &= \hat{A}^R \\ (A \bowtie \hat{A}) \cap \hat{A}' &= A^L \end{aligned}$$

One can define the other Weyl algebra $\hat{A} \bowtie A$ by interchanging in (3.1) the roles of φ and x .

If A is a weak C^* -Hopf algebra then using the Haar measure h one can make \hat{A} to be a Hilbert space with the scalar product $(\varphi, \psi) := \langle \varphi^* \psi, h \rangle$. The left regular representation of \hat{A} thus becomes a faithful $*$ -representation π , which is nothing but the GNS representation associated to the Haar state $\langle \cdot, h \rangle$. Now left and right invariance of the Haar measure implies that π can be extended both to a $*$ -representation π^R of $A \bowtie \hat{A}$ and to a $*$ -representation π^L of $\hat{A} \bowtie A$:

$$\begin{aligned} \pi^L(x)\psi &= x \rightarrow \psi & \pi^R(x)\psi &= \psi \leftarrow S^{-1}(x) \\ \pi^L(\varphi)\psi &= \varphi\psi & \pi^R(\varphi)\psi &= \varphi\psi \end{aligned}$$

Both representations turn out to be faithful. Hence Weyl algebras of weak C^* -Hopf algebras are C^* -algebras.

The projections $\pi^L(h)$ and $\pi^R(h)$ in the left regular representation project onto the subspaces \hat{A}^L and \hat{A}^R , respectively. Now using standard technics [J, GHJ] it is not difficult to show that the two Weyl algebras arise through the basic construction from the inclusions $\hat{A}^R \subset \hat{A}$ and $\hat{A}^L \subset \hat{A}$, respectively, in such a way that $\pi^L(h)$ and $\pi^R(h)$ become the Jones projections with the associated conditional expectations being given by (2.18). Utilizing also the dual statements we obtain the following Jones triples

$$\begin{aligned} A^L \subset A \subset A \rtimes \hat{A} & \quad \hat{A} \rtimes A \supset A \supset A^R \\ \hat{A}^L \subset \hat{A} \subset \hat{A} \rtimes A & \quad A \rtimes \hat{A} \supset \hat{A} \supset \hat{A}^R \end{aligned}$$

The double $\mathcal{D} = A \rtimes \hat{A}$ of a weak $*$ -Hopf algebra A can be defined by generators and relations exactly in the same way as for Hopf algebras (e.g. see Appendix B of [NSz1]). \mathcal{D} is a unital $*$ -algebra generated by symbols $\mathcal{D}(x)$, $x \in A$ and $\mathcal{D}(\varphi)$, $\varphi \in \hat{A}$. The relations require that $\mathcal{D}(A)$ and $\mathcal{D}(\hat{A})$ form unital subalgebras of \mathcal{D} isomorphic to A and \hat{A} , respectively. Furthermore the following commutation relation is postulated:

$$\mathcal{D}(\varphi)\mathcal{D}(x) = \mathcal{D}(x_{(2)})\mathcal{D}(\varphi_{(2)}) \langle \varphi_{(1)}, x_{(3)} \rangle \langle \varphi_{(3)}, S^{-1}(x_{(1)}) \rangle \quad (3.4)$$

for all $x \in A$, $\varphi \in \hat{A}$. It follows that \mathcal{D} is an amalgamated tensor product of A and \hat{A} with the common $*$ -subalgebras being $\partial A := \langle A^L, A^R \rangle$ of A and $\partial \hat{A} := \langle \hat{A}^L, \hat{A}^R \rangle$ of \hat{A} with identifications given between A^R and \hat{A}^L as in the Weyl algebra $A \rtimes \hat{A}$ and between \hat{A}^R and A^L as in the Weyl algebra $\hat{A} \rtimes A$. In other words the following identities hold in \mathcal{D} :

$$\mathcal{D}(e^i) = \mathcal{D}(\hat{e}_i^*), \quad \mathcal{D}(e_i) = \mathcal{D}(\hat{e}^{i*}), \quad i = 1, \dots, N. \quad (3.5)$$

One can introduce a coproduct $\Delta_{\mathcal{D}}$, a counit $\varepsilon_{\mathcal{D}}$, and an antipode $S_{\mathcal{D}}$ on the double just like in case of Hopf algebras. These structural maps turn out to satisfy all axioms (A.1–4). Hence the double of a weak $*$ -Hopf algebra is a weak $*$ -Hopf algebra again.

Similar statement holds also in the C^* case. As a matter of fact let A be a weak C^* -Hopf algebra and h, \hat{h} denote the Haar measures in A and \hat{A} , respectively. Then the map $\hat{\mathcal{D}} \ni \Phi \mapsto \langle \Phi, \mathcal{D}(h)\mathcal{D}(\hat{h}) \rangle$ can be shown to define a faithful positive linear functional. Thus $\hat{\mathcal{D}}$, and therefore its dual \mathcal{D} too, is a weak C^* -Hopf algebra.

Like for Hopf algebras the double is always quasitriangular. That is if $\{b^k\}$ and $\{\beta_k\}$ denote bases of A and \hat{A} that are in duality, $\langle \beta_k, b^l \rangle = \delta_k^l$, then

$$\mathcal{R} := \sum_k \mathcal{D}(b^k) \otimes \mathcal{D}(\beta_k) \in \mathcal{D} \otimes \mathcal{D} \quad (3.6)$$

is a partial isometry satisfying the usual hexagon identities and intertwines between $\Delta_{\mathcal{D}}$ and the opposite coproduct $\Delta_{\mathcal{D}}^{op}$.

A quantum chain can now be constructed using methods of [NSz1, NSz2] based on a finite dimensional weak C^* -Hopf algebra A . Its observable algebra is defined to be the infinite crossed product $\mathcal{A} = \dots A \rtimes \hat{A} \rtimes A \rtimes \hat{A} \rtimes \dots$ equipped with the following local structure. For each interval $I = \{i, i+1, \dots, j\} \subset \mathbb{Z}$ we have the local algebra $\mathcal{A}(I) \equiv \mathcal{A}_{i,j} := A_i \rtimes A_{i+1} \rtimes \dots \rtimes A_j$ with the 1-point algebras being $A_{2i} = A$, $A_{2i+1} = \hat{A}$. The 2-point algebras $\mathcal{A}_{i,i+1}$ in this net are the Weyl algebras. It follows that the net will fail to be split, i.e. $\mathcal{A}(I)$ will not be simple for any I , and will not have the intersection property, i.e. $A_i \cap A_{i+1} \neq C\mathbb{1}$ (provided A^L is not simple). The failure of these two properties can encourage one to hope that the statistical

dimensions will be non-integers. Indeed, if we define a localized coaction $\rho: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{D}$ of the double \mathcal{D} of \mathcal{A} by formula (3.18) of [NSz1] then we exhibit the double as a quantum symmetry of the superselection sectors (whether it is also universal is not clear at present). Therefore the statistical dimensions and other representation theoretic data can be determined solely from the knowledge of the double. The example treated in Sect. 5 will demonstrate that the double can really have non-integral dimensions.

Crucial is for the construction that the net \mathcal{A} satisfies Haag duality. Like for Hopf algebras Haag duality in \mathcal{A} is intimately related to the existence of integrals in \mathcal{A} and $\hat{\mathcal{A}}$. As a matter of fact if h_i denotes the unique Haar integral in \mathcal{A}_i then the maps

$$\eta_i^L(a) := h_{i(1)} a S(h_{i(2)}) , \quad \eta_i^R(a) := S(h_{i(1)}) a h_{i(2)} , \quad a \in \mathcal{A} \quad (3.7)$$

define conditional expectations satisfying a generalized η -property (cf. [NSz1]):

$$\eta_i^L(\mathcal{A}_{-\infty, i-1}) = \mathcal{A}_{-\infty, i-2} , \quad \eta_i^R(\mathcal{A}_{i+1, \infty}) = \mathcal{A}_{i+2, \infty} \quad (3.8)$$

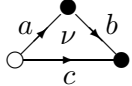
which implies, as in [NSz1], Haag duality of the net \mathcal{A} .

Field algebras can be constructed by taking crossed products with respect to actions ρ of the dual of the double: $\mathcal{F} = \mathcal{A} \rtimes_{\rho} \hat{\mathcal{D}}$ in which a local subalgebra of \mathcal{A} becomes amalgamated with the "left subalgebra" $\hat{\mathcal{D}}^L$. Although the inclusion $\mathcal{A} \subset \mathcal{F}$ in general is not irreducible, $\mathcal{A}' \cap \mathcal{F} = \hat{\mathcal{D}}^R$, the gauge principle is satisfied. Namely, \mathcal{F} carries a natural action γ of the symmetry algebra \mathcal{D} such that \mathcal{A} reappears in \mathcal{F} as the invariant subalgebra: $\gamma_{h_{\mathcal{D}}}(\mathcal{F}) = \mathcal{A}$, where $h_{\mathcal{D}}$ is the Haar integral of \mathcal{D} . In other words, in the double crossed product $\mathcal{A} \rtimes_{\rho} \hat{\mathcal{D}} \rtimes_{\gamma} \mathcal{D}$ the commutant of \mathcal{D} is the observable algebra \mathcal{A} .

4 Weak C^* -Hopf Algebras and the Pentagon Equation

As it was pointed out in [MS1] any quantum field theory determines a solution F of the pentagon equations (1.1) which in turn coincides with the recoupling coefficients (6j-symbols) of the underlying quantum symmetry. The task is then to reconstruct a quantum group from its recoupling coefficients F . Here we shall sketch an argument how a weak C^* -Hopf algebra can be reconstructed from a solution F of the pentagon equations. (The braiding structure which is also an important ingredient in QFT is neglected throughout this paper.)

We formulate the argument in a quite general combinatorial way which was strongly motivated by Ocneanu's quantum cohomology [O]. Let K be a finite simplicial 3-complex having only 2 vertices: \circ and \bullet . There are 3 types of (oriented) edges: $\bullet \xrightarrow{p} \bullet$, $\circ \xrightarrow{i} \bullet$, and $\circ \xrightarrow{\hat{p}} \circ$, but no edges go from \bullet to \circ . Faces may be attached to three edges a, b, c

if their endpoints fit into a figure of the kind  although possibly with some other

(allowed) coloring of the vertices. The faces bordered by the triangle (a, b, c) are labelled by $\nu = 1, \dots, N_{ab}^c$. (If all the three edges p, q, r connect \bullet with \bullet then the number of such faces N_{pq}^r can be thought the fusion coefficients of a quantum field theory or of a monoidal category, although we do not want to make any restriction on these numbers.) One 3-simplex is attached to four faces whenever they form the boundary of a tetrahedron.

Now we construct two C^* -algebras: \mathcal{A} associated to the vertex \bullet and $\hat{\mathcal{A}}$ associated to \circ . Each edge $\bullet \xrightarrow{q} \bullet$ corresponds to a simple block of \mathcal{A} . The simple block q is by definition

common edge of type $\bullet \longrightarrow \bullet$ with the other two vertices being white. A matrix unit of \hat{A} on the other hand is a pair of triangles (4.2) with $\circ \longrightarrow \circ$ as the common edge and the other two vertices being black. Hence rotating one of them by 180° around its "main diagonal" we can glue them together to obtain a tetrahedron with exactly 2 white vertices. Hence the pairing will be defined in terms of a 3-chain F_2 (an "Ocneanu cell") so that

$$\left\langle \begin{bmatrix} i' & i \\ j' & j \end{bmatrix}_{\hat{p}}, \begin{pmatrix} k' & k \\ l' & l \end{pmatrix}_q \right\rangle = \delta_{i'k'} \delta_{j'k} \delta_{i,l'} \delta_{j,l} \cdot \begin{array}{c} \bullet \\ \nearrow q \\ \circ \xrightarrow{\hat{p}} \circ \searrow k \\ \nwarrow k' \bullet \\ \searrow l' \bullet \end{array} \quad (4.3)$$

The condition for this pairing to transpose the coproduct of A to the product of \hat{A} is again a pentagon of the type (P_2) : $F_2 F_1 F_1 = F_2 F_2$. Similarly, the compatibility of the product on A with the coproduct on \hat{A} is the pentagon equation (P_3) : $F_2 F_3 F_3 = F_2 F_2$.

Summarizing, if $F = (F_0, F_1, F_2, F_3, F_4)$ is a unitary 3-cocycle on K , i.e. satisfies the Big Pentagon equation $(P) = (P_0, P_1, P_2, P_3, P_4, P_5)$ and unitarity, then its restrictions F_n $n = 0, \dots, 4$ to tetrahedra with fixed coloring of the vertices determine the various structural maps of a weak C^* -Hopf algebra A . We recall that once having the C^* -algebras A and \hat{A} in duality the antipode can be introduced by the formula $S := * \circ *$ where the antilinear involution $*$ is defined by $\langle \varphi, x_* \rangle := \overline{\langle \varphi^*, x \rangle}$. The resulting weak C^* -Hopf algebra will be pure iff the complex K was 2-connected. For such pure quantum groups A we have derived from the axioms the existence of a unique block $q = 0$, the trivial representation, a unique involution $q \mapsto \bar{q}$ of the blocks of A determined by the action of the antipode on the center, and one such involution $\hat{p} \mapsto \bar{\hat{p}}$ of the dual blocks. Therefore the fact that the fusion ring determined by the fusion coefficients N_{ab}^c is associative, unital, and involutive are not assumptions on the complex K but all are consequences of the existence of a solution for the Big Pentagon equation.

In practice, however, we have only given a solution F_0 of (P_0) . That is F is known only on the subcomplex $K_A \subset K$ containing the single vertex \bullet . There seems to be a canonical way to extend F_0 to a solution F determining a selfdual weak C^* -Hopf algebra A , i.e. such that $\hat{A} \cong A$ as weak C^* -Hopf algebras. In order to have such an extension, however, we need to postulate — on K_A at least — the existence of the trivial block 0 and the involution $q \mapsto \bar{q}$ with the usual properties

$$N_{0b}^c = \delta_{bc} = N_{b0}^c \quad (4.4a)$$

$$N_{ab}^0 = \delta_{a\bar{b}} \quad (4.4b)$$

$$N_{c\bar{b}}^a = N_{ab}^c = N_{\bar{a}c}^b \quad (4.4c)$$

Now the extension goes as follows. Define K by adjoining to K_A another copy of it denoted $K_{\hat{A}}$ with its vertex denoted by \circ and then connect the two vertices by as many edges as K_A has. Therefore all three types of edges of K will carry the same label p running over the set of sectors (of A or of the underlying quantum field theory). Triangles with mixed vertices are set between three edges (p, q, r) whenever there is a triangle between the edges of the same labels in K_A . With this choice we obtain $A \cong \hat{A} \cong \oplus_p M_{n_p}$ where $n_p = \sum_{q,r} N_{pq}^r$. Now the extension F is obtained by setting $F_1 = F_3 = F_4 = F_0$ and F_2 to be the inverse of F_0 in the sense of the equation

$$\sum_r F_2 \binom{pqr}{s}_{tu} F_0 \binom{p'qr}{s}_{tu} = \delta^{pp'} \cdot \text{constraints} \quad (4.5)$$

for each fixed value of q, s, t, u . The existence of this kind of inverse (and therefore of the pairing) follows since (4.4abc) imply that F_0 has an S_4 symmetry [FGV2] hence unitarity implies its invertibility also in the required indices.

Therefore a selfdual unimodular finite quantum group can be associated to any rational quantum field theory describing its internal symmetry on the level of representation categories. Of course, this does not solve the problem of how to determine unique integers n_q [S], i.e. the problem of uniqueness of the quantum symmetry. There may be other extensions K of K_A and solutions F on K of the Big Pentagon equation extending the given F_0 . Thus there may be other weak C^* -Hopf algebras B (though perhaps not selfdual) such that $\mathbf{Rep} B$ is equivalent to $\mathbf{Rep} A$.

5 An Example with Lee–Yang Fusion Rules

Consider the fusion ring generated by two selfconjugate sectors 0 and 1 where 0 is the unit element and $1 \times 1 = 0 + 1$. Corresponding to these fusion rules there is a solution F of the pentagon equation [FGV1].

The weak C^* -Hopf algebra $A_{L.Y.}$ given below has been constructed from that solution using the general method of Sect. 4. Setting the free parameter ζ in [FGV1] to 1 all structural maps can be written as integer polynomials in $z = \sqrt{(\sqrt{5} - 1)/2}$. The intrinsic dimensions of the two sectors are $d_0 = 1$ and $d_1 = z^{-2} \equiv (\sqrt{5} + 1)/2$.

As a C^* -algebra $A_{L.Y.} = M_2 \oplus M_3$. We fix matrix units e_0^{ij} in M_2 and e_1^{ij} in M_3 . The coproduct is given by

$$\begin{aligned}
\Delta(e_0^{11}) &= e_0^{11} \otimes e_0^{11} + e_1^{11} \otimes e_1^{33} \\
\Delta(e_0^{12}) &= e_0^{12} \otimes e_0^{12} + z^2 e_1^{13} \otimes e_1^{31} + z e_1^{12} \otimes e_1^{32} \\
\Delta(e_0^{22}) &= e_0^{22} \otimes e_0^{22} + z^4 e_1^{33} \otimes e_1^{11} + z^3 e_1^{32} \otimes e_1^{12} + z^3 e_1^{23} \otimes e_1^{21} + z^2 e_1^{22} \otimes e_1^{22} \\
\Delta(e_1^{11}) &= e_0^{11} \otimes e_1^{11} + e_1^{11} \otimes e_0^{22} + e_1^{11} \otimes e_1^{22} \\
\Delta(e_1^{12}) &= e_0^{12} \otimes e_1^{12} + e_1^{12} \otimes e_0^{22} + z e_1^{13} \otimes e_1^{21} - z^2 e_1^{12} \otimes e_1^{22} \\
\Delta(e_1^{13}) &= e_0^{12} \otimes e_1^{13} + e_1^{13} \otimes e_0^{21} + e_1^{12} \otimes e_1^{23} \\
\Delta(e_1^{22}) &= e_0^{22} \otimes e_1^{22} + e_1^{22} \otimes e_0^{22} \\
&\quad + z^2 e_1^{33} \otimes e_1^{11} - z^3 e_1^{32} \otimes e_1^{12} - z^3 e_1^{23} \otimes e_1^{21} + z^4 e_1^{22} \otimes e_1^{22} \\
\Delta(e_1^{23}) &= e_0^{22} \otimes e_1^{23} + e_1^{23} \otimes e_0^{21} + z e_1^{32} \otimes e_1^{13} - z^2 e_1^{22} \otimes e_1^{23} \\
\Delta(e_1^{33}) &= e_0^{22} \otimes e_1^{33} + e_1^{33} \otimes e_0^{11} + e_1^{22} \otimes e_1^{33}
\end{aligned}$$

The counit and the antipode are as follows:

$$\begin{aligned}
\varepsilon(e_0^{ij}) &= 1 \quad i, j = 1, 2, & \varepsilon(e_1^{ij}) &= 0 \quad i, j = 1, 2, 3, \\
S(e_0^{ij}) &= e_0^{ji} \quad i, j = 1, 2 & S(e_1^{ij}) &= z^{i-j} e_1^{\bar{j}\bar{i}} \quad i, j = 1, 2, 3,
\end{aligned}$$

where we introduced the notation $\bar{1} = 3$, $\bar{2} = 2$, $\bar{3} = 1$. One can check easily that they satisfy the axioms (A1–4). The left and right subalgebras turn out to be Abelian, isomorphic to $M_1 \oplus M_1$, with orthonormal bases given by the minimal projections

$$\begin{aligned}
e^1 &= e_0^{11} + e_1^{11}, & e^2 &= e_0^{22} + e_1^{22} + e_1^{33} \\
e_1 &= e_0^{11} + e_1^{33}, & e_2 &= e_0^{22} + e_1^{11} + e_1^{22}
\end{aligned}$$

The projections $\Pi^{L/R}$ take the following simple form

$$\begin{aligned}\Pi^L(e_0^{ij}) &= e^i & \Pi^R(e_0^{ij}) &= e_j \\ \Pi^L(e_1^{ij}) &= 0 & \Pi^R(e_1^{ij}) &= 0\end{aligned}$$

The Haar measure h is a rank 1 projection belonging to the support of ε , i.e. of the trivial block 0:

$$h = \frac{1}{2} \sum_{i,j=1,2} e_0^{ij}.$$

The above weak C^* -Hopf algebra is obviously the smallest in dimension among the weak C^* -Hopf algebras with $S^2 \neq \text{id}$. Nevertheless computing the structural maps of its double is quite a horrible task. Since the existence of a double with non-integer intrinsic dimensions is crucial for the construction of quantum chains with non-integer statistical dimensions, we have calculated — using some computer aid — the block structure, the fusion rules, and the dimensions d_p of the double $\mathcal{D}_{L,Y}$ of the Lee-Yang quantum group $A_{L,Y}$. The results are the following. As an algebra $\mathcal{D}_{L,Y}$ is isomorphic to

$$\mathcal{D}_{L,Y} = M_2 \oplus M_3 \oplus M_3 \oplus M_5$$

All the four sectors are selfconjugate. Denoting these sectors by $2, 3, 3', 5$, respectively we have the following fusion rules. 2 is the trivial representation, therefore $2 \times p = p \times 2 = p$ for all sectors p . The fusion is commutative: $p \times q = q \times p$ for all p, q . Furthermore

$$\begin{aligned}3 \times 3 &= 2 + 3 & 3' \times 3' &= 2 + 3' & 5 \times 5 &= 2 + 3 + 3' + 5 \\ 3 \times 3' &= 5 & 3' \times 5 &= 3 + 5 \\ 3 \times 5 &= 3' + 5\end{aligned}$$

These fusion rules yield the intrinsic dimensions $d_2 = 1$, $d_3 = d_{3'} = z^{-2}$, $d_5 = z^{-4}$. Except d_2 these are not integers and are related to the d_1 of $A_{L,Y}$ in a very simple way, $d_2 = d_1^0$, $d_3 = d_1$, $d_{3'} = d_1$, $d_5 = d_1^2$ which may have its explanation in the selfduality of $A_{L,Y}$.

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Appendix: Blowing up the Quasi-Double $\mathcal{D}^\omega(G)$

Let G be a finite group of order $|G| = N$ and ω a $U(1)$ valued 3-cocycle on G . Then $\mathcal{D}^\omega(G)$ is the quasi-Hopf algebra with product and coproduct given on the basis elements $(g, h) \in G \times G$ as follows [DPR]:

$$\begin{aligned}(g, h)(g', h') &= \delta_{gh, hg'} \theta_g(h, h') \cdot (g, hh') \\ \text{where } \theta_g(h, h') &= \frac{\omega(h, h', h'^{-1}h^{-1}ghh') \omega(g, h, h')}{\omega(h, h^{-1}gh, h')}\end{aligned}$$

$$\begin{aligned}\Delta_\omega((g, h)) &= \sum_{k \in G} \gamma_h(k, k^{-1}g) \cdot (k, h) \otimes (k^{-1}g, h) \\ \text{where} \quad \gamma_h(x, y) &= \frac{\omega(h, h^{-1}xh, h^{-1}yh) \omega(x, y, h)}{\omega(x, h, h^{-1}yh)}\end{aligned}$$

Δ_ω is quasi-coassociative and no skrooched version $\Delta'_\omega = \text{Ad}_u \circ \Delta_\omega$ of it can be coassociative, unless ω is a coboundary.

Consider the full matrix algebra M_N with a fixed choice of matrix units $\{e_{ab}\}$ and make it a weak C^* -Hopf algebra by introducing $\Delta_M(e_{ab}) = e_{ab} \otimes e_{ab}$, $\varepsilon_M(e_{ab}) = 1$, $S_M(e_{ab}) = e_{ba}$ for $a, b = 1, \dots, N$. Now we claim that on the tensor product $\mathcal{M}^\omega(G) = \mathcal{D}^\omega(G) \otimes M_N$, which is a weak quasi-Hopf algebra, there is a skrooching $U \in \mathcal{M}^\omega(G) \otimes \mathcal{M}^\omega(G)$ transforming the quasi-coassociative non-unital coproduct $\Delta_\omega \otimes \Delta_M$ into a coassociative non-unital coproduct Δ :


$$\begin{aligned}\Delta(\xi \otimes m) &= U \cdot (\Delta_\omega \otimes \Delta_M)(\xi \otimes m) \cdot U^* \\ U &= \sum_{a, b, c \in G} \omega^{-1}(a, b, c) [(c, 1) \otimes e_{ab, a}] \otimes [(b, 1) \otimes e_{a, a}] .\end{aligned}$$

The counit remains the tensor product $\varepsilon = \varepsilon_\omega \otimes \varepsilon_M$ while the antipode undergoes a skrooching $S = \text{Ad}_V \circ (S_\omega \otimes S_M)$ where $V = U_1 \cdot (S_\omega \otimes S_M)(U_2)$. With the structure maps (Δ, ε, S) the blown up double $\mathcal{M}^\omega(G)$ becomes a weak C^* -Hopf algebra. It is also quasitriangular with R-matrix $U^{op}(R_1 \otimes I \otimes R_2 \otimes I)U^*$ where $R = R_1 \otimes R_2$ denotes the old R-matrix of the quasi-double. This blown up double and the original quasi-Hopf algebra have identical representation theories, namely $\mathbf{Rep} \mathcal{M}^\omega(G)$ and $\mathbf{Rep} \mathcal{D}^\omega(G)$ are equivalent as braided monoidal categories. $\mathcal{M}^\omega(G)$ depends non-trivially on the cohomology class of ω although it is coassociative for any choice of the 3-cocycle.

$\mathcal{M}^\omega(G)$ can be shown to be the double in the sense of Sect. 3 of a weak C^* -Hopf algebra $A^\omega(G)$. The latter one can be defined as a blowing up of the Hopf algebra $\mathcal{C}(G)$ of complex functions on G . If δ_g , $g \in G$ denote the minimal projections in $\mathcal{C}(G)$ then $A^\omega(G)$ is obtained from the tensor product weak Hopf algebra $\mathcal{C}(G) \otimes M_N$ by skrooching with the partial isometry

$$u = \sum_{a, b, c \in G} \omega^{-1}(a, b, c) [\delta_b \otimes e_{a, a}] \otimes [\delta_c \otimes e_{ab, a}] .$$

Equivalently $A^\omega(G)$ can be constructed by the method of Sect. 4 from the cocycle ω itself, if it is interpreted as a solution $F = \omega^{-1}$ of the pentagon equation on an appropriate complex $K_A = K(G)$. As a matter of fact, the complex $K(G)$ has one vertex \bullet and N edges labelled by

$g \in G$. A face is attached to the triangle  if and only if the zero curvature condition

$gh = k$ is satisfied. The resulting C^* -algebra $A = A^\omega(G)$ is isomorphic to $\mathcal{C}(G) \otimes M_N$ and its Wigner and Racah coefficients, as well as its Ocneanu cell, are given by the cocycle ω . The relation between the two constructions can be elucidated by the formula

$$\delta_g \otimes e_{ab} = \begin{array}{c} \bullet \\ \swarrow \quad \searrow \\ \circ \quad \quad \circ \\ \swarrow \quad \searrow \\ \bullet \end{array} \begin{array}{l} a \quad b \\ g \quad g \\ ag \quad bg \end{array}$$

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